5.12. **Integration by parts for some Itô integrals.** Let $g \in \mathcal{C}^2(\mathbb{R})$ and $(B_t, t \geq 0)$, a standard Brownian motion.

(a) Use Itô's formula to prove that for any $t \geq 0$

$$\int_0^t g(s)\,\mathrm{d}B_s = g(t)B_t - \int_0^t B_s g'(s)\,\mathrm{d}s.$$ [Primijeniti Itovu formulu na $f(t,B_t)=xg(B_t)$]

(b) process given by

$$X_t = t^2 B_t - 2\int_0^t sB_s\,\mathrm{d}s$$

Izračunati $E[X(t)]$ i $E[X^2(t)]$.

Exercise 5.5 Suppose that X satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

Now define Y by $Y_t = X_t^\beta$, where β is a real number. Then Y is also a GBM process. Compute dY and find out which SDE Y satisfies.

Koristeći istu ideju kao kod heurističkog izvođenja rješenja stohastičke diferencijalne jednačine geometrijskog Braunovog kretanja, riješiti sljedeću uopšteniju jednačinu:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad S(t) > 0$$

Formulisati Itovu formulu *f(B_t)* (gdje je *f* dva puta diferencijabilna funkcija), a zatim je primijeniti na rješavanje sljedećeg zadatka:

4.5. Let $B_t \in \mathbf{R}$, $B_0 = 0$. Define

$$\beta_k(t) = E[B_t^k] \; ; \qquad k = 0, 1, 2, \ldots ; \;\; t \geq 0 \; .$$

Use Itô's formula to prove that

$$\beta_k(t) = \tfrac{1}{2} k(k-1) \int_0^t \beta_{k-2}(s) ds \; ; \qquad k \geq 2 \; .$$

Deduce that

$$E[B_t^4] = 3t^2 \qquad \text{(see (2.2.14))}$$

and find

$$E[B_t^6] \; .$$

Neka je
$$dX(t) = a_X(t)dt + b_Y(t)dW(t)$$
$$dY(t) = a_Y(t)dt + b_Y(t)dW(t)$$

Primjenjući (jednodimenzionu) Itovu formulu na procese *X^2, Y^2 $(X+Y)^2$* riješiti sljedeći zadatak:

Let X_t, Y_t be Itô processes in $\mathbf{R}$. Prove that

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \ .$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s \ .$$

Neka je $Z(t) = \int_0^t g(s)dW(s)$, where g is an adapted stochastic process.

Izračunati *$dZ(t)$*

Neka je

$$dX(t) = a_X(t)dt + b_Y(t)dW(t)$$
$$dY(t) = a_Y(t)dt + b_Y(t)dW(t)$$

Formulisati 2-dimenzionu Itovu formulu, a zatim je primijeniti funkciju *f(x,y)=xy* za rješavanje sljedećeg zadatka:

Let X_t, Y_t be Itô processes in $\mathbf{R}$. Prove that

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \ .$$

Deduce the following general *integration by parts formula*

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s \ .$$

(Exponential martingales)

Suppose $\theta\ (t,\omega) \in \mathcal{V}[0,T]$ where $T \leq \infty$. Define

$$Z_t = \exp\left\{ \int_0^t \theta(s,\omega)dB(s) - \tfrac{1}{2} \int_0^t \theta^2(s,\omega)ds \right\}; \qquad 0 \leq t \leq T$$

where $B(s) \in \mathbf{R}$

a) Use Itô's formula to prove that

$$dZ_t = Z_t \theta(t,\omega) dB(t)\ .$$

b) Deduce that Z_t is a martingale for $t \leq T$, provided that

$$Z_t \theta_k(t,\omega) \in \mathcal{V}[0,T] \qquad \text{for } 1 \leq k \leq n\ .$$

4.9 Exercises

Exercise 4.1 Compute the stochastic differential dZ when

(c) $Z(t) = e^{\alpha W(t)}$

(d) $Z(t) = e^{\alpha X(t)}$, where X has the stochastic differential

$$dX(t) = \mu dt + \sigma dW(t)$$

(μ and σ are constants).

U oba slučaja izračunati *E[Z(t)] i E[Z²(t)].*

Exercise 4.4 Suppose that X has the stochastic differential

$$dX(t) = \alpha X(t)dt + \sigma(t)dW(t),$$

where α is a real number whereas $\sigma(t)$ is any stochastic process. Use the technique in Example 4.13 in order to determine the function $m(t) = E[X(t)]$.

Ako $\sigma(t)=\sigma$ konstantna funkcija, riješiti datu jednačinu primjenom Itove formule na $f(t,X(t))=\exp(-\alpha t)X(t)$.

Neka je $dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t).$

Primjenjujući Itovu formulu na *f(t,R(t))=* $e^{\beta t}R(t)$ dokazati jednakost:

$$e^{\beta t}R(t) = R(0) + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma \int_0^t e^{\beta u}\sqrt{R(t)}dW(u).$$

Izračunati $E[R(t)]$ i $E[R^2(t)]$.

Neka je $dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$. Here α, β and σ are positive constants.

Primjenjujući Itovu formulu na $f(t,R(t)) = e^{\beta t} R(t)$ izvesti izraz za $R(t)$.

Izračunati $E[R(t)]$ i $E[R^2(t)]$.